

HOMOTOPY DIMENSION OF ORBITS OF MORSE FUNCTIONS ON SURFACES

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ABSTRACT. Let M be a compact surface, P be either the real line \mathbb{R} or the circle S^1 , and $f : M \rightarrow P$ be a C^∞ Morse map. The identity component $\mathcal{D}_{\text{id}}(M)$ of the group of diffeomorphisms of M acts on the space $C^\infty(M, P)$ by the following formula: $h \cdot f = f \circ h^{-1}$ for $h \in \mathcal{D}_{\text{id}}(M)$ and $f \in C^\infty(M, P)$. Let $\mathcal{O}(f)$ be the orbit of f with respect to this action and n be the total number of critical points of f . In this note we show that $\mathcal{O}(f)$ is homotopy equivalent to a certain covering space of the n -th configuration space of the interior $\text{Int}M$. This in particular implies that the (co-)homology of $\mathcal{O}(f)$ vanish in dimensions greater than $2n - 1$, and the fundamental group $\pi_1 \mathcal{O}(f)$ is a subgroup of the n -th braid group $\mathcal{B}_n(M)$.

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1. INTRODUCTION

Let M be a compact surface, P be either the real line \mathbb{R} or the circle S^1 . Then the group $\mathcal{D}(M)$ of C^∞ diffeomorphisms of M acts on the space $C^\infty(M, P)$ by the following formula:

$$(1.1) \quad h \cdot f = f \circ h^{-1}$$

for $h \in \mathcal{D}(M)$ and $f \in C^\infty(M, P)$.

We say that a smooth (C^∞) map $f : M \rightarrow P$ is *Morse* if

- (i) critical points of f are non-degenerate and belong to the interior of M ;
- (ii) f is constant on every connected component of ∂M .

Let $f \in C^\infty(M, P)$, Σ_f be the set of critical points of f , and $\mathcal{D}(f, \Sigma_f)$ be the subgroup of $\mathcal{D}(M)$ consisting of diffeomorphisms h such that $h(\Sigma_f) = (\Sigma_f)$.

Then we can define the stabilizers $\mathcal{S}(f)$ and $\mathcal{S}(f, \Sigma_f)$, and orbits $\mathcal{O}(f)$ and $\mathcal{O}(f, \Sigma_f)$ with respect to the actions of the groups $\mathcal{D}(M)$

and $\mathcal{D}(f, \Sigma_f)$. Thus

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) : f \circ h^{-1} = f\}, \quad \mathcal{O}(f) = \{f \circ h^{-1} : h \in \mathcal{D}(M)\},$$

$$\mathcal{S}(f, \Sigma_f) = \mathcal{S}(f) \cap \mathcal{D}(f, \Sigma_f).$$

We endow the spaces $\mathcal{D}(M)$ and $C^\infty(M, P)$ with the corresponding C^∞ Whitney topologies. They induce certain topologies on the stabilizers and orbits.

Let $\mathcal{D}_{\text{id}}(M)$ and $\mathcal{D}_{\text{id}}(f, \Sigma_f)$ be the identity path components of the groups $\mathcal{D}(M)$ and $\mathcal{D}(f, \Sigma_f)$, $\mathcal{S}_{\text{id}}(f)$ and $\mathcal{S}_{\text{id}}(f, \Sigma_f)$ be the identity path components of the corresponding stabilizers, and $\mathcal{O}_f(f)$ and $\mathcal{O}_f(f, \Sigma_f)$ be the path-components of f in the corresponding orbits with respect to the induced topologies.

Lemma 1. *If Σ_f is discrete set, e.g. when f is Morse, then $\mathcal{S}_{\text{id}}(f, \Sigma_f) = \mathcal{S}_{\text{id}}(f)$.*

Proof. Since $\mathcal{S}(f, \Sigma_f) \subset \mathcal{S}(f)$, we have that $\mathcal{S}_{\text{id}}(f, \Sigma_f) \subset \mathcal{S}_{\text{id}}(f)$. Conversely, let $h_t : M \rightarrow M$ be an isotopy such that $h_0 = \text{id}_M$ and $h_t \in \mathcal{S}(f)$ for all $t \in I$, i.e. $f \circ h_t = f$. We have to show that $h_t \in \mathcal{S}(f, \Sigma_f)$ for all $t \in I$. Notice that $d(f \circ h_t) = h_t^* df = df$, whence $h_t(\Sigma_f) = \Sigma_f$. Since Σ_f is discrete and $h_0 = \text{id}_M$ fixes Σ_f , we see that so does every h_t , i.e. $h_t \in \mathcal{S}(f, \Sigma_f)$. \square

Let $f : M \rightarrow P$ be a Morse map. Denote by c_i , ($i = 0, 1, 2$), the total numbers of critical points of f of index i and let $n = c_0 + c_1 + c_2$ be the total number of critical points of f .

Notice that for every Morse map f its orbits $\mathcal{O}(f)$ and $\mathcal{O}(f, \Sigma_f)$ are Fréchet submanifolds of $C^\infty(M, P)$ of finite codimension, see [4, 5]. Therefore, e.g. [3], these orbits have the homotopy types of CW-complexes. But in general these complexes may have infinite dimensions.

Let X be a topological space which is homotopy equivalent to some CW-complex. Then a *homotopy dimension* h.d. X of X is the minimal dimension of a CW-complex homotopy equivalent to X . In particular h.d. X can be equal to ∞ . It is also evident that if h.d. $X < \infty$, then (co-)homology of X vanish in dimensions greater than h.d. X .

If π is a finitely presented group π , then the *geometric dimension* of π , denoted g.d. π , is the homotopy dimension of its Eilenberg-Mac Lane space $K(\pi, 1)$:

$$\text{g.d. } \pi := \text{h.d. } K(\pi, 1).$$

In [2, Theorems 1.3, 1.5, 1.9] the author described the homotopy types of $\mathcal{S}_{\text{id}}(f)$, $\mathcal{O}_f(f)$, and $\mathcal{O}_f(f, \Sigma_f)$. It follows from these results

that

$$\text{h.d. } \mathcal{S}_{\text{id}}(f), \quad \text{h.d. } \mathcal{O}_f(f, \Sigma_f) \leq 1.$$

In fact, $\mathcal{S}_{\text{id}}(f)$ is contractible provided either f has at least one critical point of index 1, i.e., $c_1 \geq 1$ or M is non-orientable. Otherwise $\mathcal{S}_{\text{id}}(f) \simeq S^1$.

Also, $\mathcal{O}_f(f) \simeq S^1$ for Morse mappings $T^2 \rightarrow S^1$ and $K^2 \rightarrow S^1$ without critical points, and $\mathcal{O}_f(f)$ is contractible in all other cases, where K stands for the Klein bottle.

For $\mathcal{O}_f(f)$ the description is not so complete. But if f is *generic*, i.e., it takes distinct values at distinct critical points, then

$$\text{h.d. } \mathcal{O}_f(f) \leq \max\{c_0 + c_2 + 1, c_1 + 2\} < \infty.$$

Actually, in this case $\mathcal{O}_f(f)$ is either contractible or homotopy equivalent to T^k or to $\mathbb{R}P^3 \times T^k$ for some $k \geq 0$, where T^k is a k -dimensional torus.

Thus the upper bound for $\text{h.d. } \mathcal{O}_f(f)$ (at least in generic case) depends only on the number of critical points of f at each index.

In this note we will show that $\text{h.d. } \mathcal{O}_f(f) \leq 2n - 1$ for arbitrary Morse mapping $f : M \rightarrow P$ having exactly $n \geq 1$ critical points. Notice that if $n = 0$, then f is generic, and in fact $\text{h.d. } \mathcal{O}_f(f) \leq 1$, see [2, Table 1.10].

Theorem 2. *Let $f : M \rightarrow P$ be a Morse map and n be the total number of critical points of f . Assume that $n \geq 1$. Denote by $\mathcal{F}_n(\text{Int}M)$ the configuration space of n points of the interior $\text{Int}M$ of M . Then $\mathcal{O}_f(f)$ is homotopy equivalent to a certain covering space $\mathcal{F}(f)$ of $\mathcal{F}_n(\text{Int}M)$.*

Corollary 3. *$\text{h.d. } \mathcal{O}_f(f) \leq 2n - 1$, whence (co-)homology of $\mathcal{O}_f(f)$ vanish in dimensions $\geq 2n$.*

Proof. Since $\mathcal{F}_n(\text{Int}M)$ and its connected covering spaces are *open* manifolds of dimension $2n$, they are homotopy equivalent to CW-complexes of dimensions not greater than $2n - 1$. \square

For simplicity denote $\pi = \pi_1 \mathcal{O}_f(f)$. Since the covering map $\mathcal{F}(f) \rightarrow \mathcal{F}_n(\text{Int}M)$ yields a monomorphism of fundamental groups, we obtain the following:

Corollary 4. *The fundamental group π of $\mathcal{O}_f(f)$ is a subgroup of the n -th braid group $\mathcal{B}_n(M) = \pi_1(\mathcal{F}_n(\text{Int}M))$ of M .*

Corollary 5. *Suppose that M is aspherical, i.e., $M \neq S^2, \mathbb{R}P^2$. Then $\mathcal{O}_f(f)$ is aspherical as well, i.e., $K(\pi, 1)$ -space, whence $\text{g.d. } \pi \leq 2n - 1$.*

Proof. Actually the asphericity of $\mathcal{O}_f(f)$ for the case $M \neq S^2, \mathbb{R}P^2$ is proved in [2, Theorems 1.5, 1.9].

But it can be shown by another arguments. It is well known and can easily be deduced from [1] that for an aspherical surface M every of its configuration spaces $\mathcal{F}_n(\text{Int}M)$ and thus every covering space of $\mathcal{F}_n(\text{Int}M)$ are aspherical as well. Hence so is $\mathcal{F}(f)$ and thus $\mathcal{O}_f(f)$ itself. \square

A presentation for π will be given in another paper.

2. ORBITS OF THE ACTIONS OF $\mathcal{D}_{\text{id}}(M)$ AND $\mathcal{D}_{\text{id}}(f, \Sigma_f)$

Proposition 6. *Let $f : M \rightarrow P$ be a Morse map and*

$$(2.1) \quad p : \mathcal{D}(M) \mapsto \mathcal{O}(f), \quad p(h) = f \circ h^{-1}$$

be the natural projection. Then $\mathcal{O}_f(f)$ is the orbit of f with respect to $\mathcal{D}_{\text{id}}(M)$ and $\mathcal{O}_f(f, \Sigma_f)$ is the orbit of f with respect to $\mathcal{D}_{\text{id}}(f, \Sigma_f)$. In other words,

$$p(\mathcal{D}_{\text{id}}(M)) = \mathcal{O}_f(f) \quad \text{and} \quad p(\mathcal{D}_{\text{id}}(f, \Sigma_f)) = \mathcal{O}_f(f, \Sigma_f).$$

Proof. The proof is based on the following general statement. Let G be a topological group transitively acting on a topological space O and $f \in O$. Denote by G_e the *path-component* of the unit e in G and let O_f be the *path-component* of f in O .

Lemma 7. *Suppose that the mapping $p : G \rightarrow O$ defined by*

$$p(\gamma) = \gamma \cdot f, \quad \forall \gamma \in G$$

satisfies a covering path axiom (in particular, this holds when p is a locally trivial fibration). Then O_f is the orbit of f with respect to the induced action of G_e on O , i.e., $p(G_e) = O_f$.

Proof. Evidently, $p(G_e) \subset O_f$. Conversely, let $g \in O_f$. Then there exists a path $\omega : I \rightarrow O_f$ between f and g , i.e., $\omega(0) = f$ and $\omega(1) = g$. Since p satisfies the covering path axiom, ω lifts to the path $\tilde{\omega} : I \rightarrow G$ such that $\tilde{\omega}(0) = e$ and $\omega = p \circ \tilde{\omega}$. Then $g = \omega(1) = p \circ \tilde{\omega}(1) \in p(G_e)$. Thus $p(G_e) = O_f$. \square

It remains to note that the mapping (2.1) is a locally trivial fibration, see e.g. [4, 5], and $\mathcal{D}(M)$ (resp. $\mathcal{D}(f, \Sigma_f)$) transitively acts on the orbit $\mathcal{O}(f)$ (resp. $\mathcal{O}(f, \Sigma_f)$). Therefore the conditions of Lemma 7 are satisfied. \square

3. PROOF OF THEOREM 2

Let $\mathcal{F}_n(\text{Int}M)$ be the configuration space of n points of the interior $\text{Int}M$ of M . Thus

$$(3.1) \quad \mathcal{F}_n(\text{Int}M) = \mathcal{P}_n(\text{Int}M)/\mathbb{S}_n,$$

where

$$\mathcal{P}_n(\text{Int}M) = \{(x_1, \dots, x_n) \mid x_i \in \text{Int}M \text{ and } x_i \neq x_j \text{ for } i \neq j\}$$

is called the *pure* n -th configuration space of $\text{Int}M$, and \mathbb{S}_n is the symmetric group of n symbols freely acting on $\mathcal{P}_n(\text{Int}M)$ by permutations of coordinates.

We can regard $\mathcal{F}_n(\text{Int}M)$ as the space of n -tuples of mutually distinct points of $\text{Int}M$.

Denote by $\Sigma_f = \{x_1, \dots, x_n\}$ the set of critical points of f . Then for every $g \in \mathcal{O}_f(f)$ the set Σ_g of its critical points is a point in $\mathcal{F}_n(\text{Int}M)$. Hence the correspondence $g \mapsto \Sigma_g$ is a well-defined mapping

$$k : \mathcal{O}_f(f) \rightarrow \mathcal{F}_n(\text{Int}M), \quad k(g) = \Sigma_g.$$

Lemma 8. (i) *The mapping k is a locally trivial fibration. The connected component of the fiber containing f is homeomorphic to $\mathcal{O}_f(f, \Sigma_f)$.*

(ii) *Let $k_i : \pi_i(\mathcal{O}_f(f), f) \rightarrow \pi_i(\mathcal{F}_n(\text{Int}M), \Sigma_f)$, ($i \geq 1$), be the corresponding homomorphism of homotopy groups induced by k . Then k_1 is a **monomorphism** and all other k_i for $i \geq 2$ are isomorphisms.*

Assuming that Lemma 8 is proved we will now complete our theorem. Let $\mathcal{F}(f)$ be the covering space of $\mathcal{F}_n(\text{Int}M)$ corresponding to the subgroup

$$\pi_1 \mathcal{O}_f(f) \approx k_1(\pi_1 \mathcal{O}_f(f)) \subset \pi_1 \mathcal{F}_n(\text{Int}M).$$

Then k lifts to the mapping $\hat{k} : \mathcal{O}_f(f) \rightarrow \mathcal{F}(f)$ which induces isomorphism of all homotopy groups. Since $\mathcal{O}_f(f)$ and $\mathcal{F}(f)$ are connected, we obtain from (2) that \hat{k} is a desired homotopy equivalence. Theorem 2 is proved modulo Lemma 8.

Proof of Lemma 8. (i) Recall, [1], that the following *evaluation* map

$$e : \mathcal{D}_{\text{id}}(M) \rightarrow \mathcal{F}_n(\text{Int}M), \quad e(h) = h(\Sigma_f)$$

is a locally trivial principal fibration with fiber

$$\hat{\mathcal{D}}(f) = \mathcal{D}_{\text{id}}(M) \cap \mathcal{D}(f, \Sigma_f).$$

Let $p : \mathcal{D}_{\text{id}}(M) \rightarrow \mathcal{O}_f(f)$ be the projection defined by $p(h) = f \circ h^{-1}$. Then the set of critical points of the function $f \circ h^{-1} \in \mathcal{O}_f(f)$ is $h(\Sigma_f)$. Therefore e coincides with the following composition:

$$e = k \circ p : \mathcal{D}_{\text{id}}(M) \xrightarrow{p} \mathcal{O}_f(f) \xrightarrow{k} \mathcal{F}_n(\text{Int}M).$$

Since e and (by Proposition 6) the mapping p are principal locally trivial fibrations, we obtain that k is also a locally trivial fibration with fiber $\hat{\mathcal{O}}(f)$ being the orbit of f with respect to the group $\hat{\mathcal{D}}(f)$.

It is easy to see that the identity component of the group $\hat{\mathcal{D}}(f)$ coincides with $\mathcal{D}_{\text{id}}(f, \Sigma_f)$, whence by Proposition 6, the connected component of $\hat{\mathcal{O}}(f)$ containing f is $\mathcal{O}_f(f, \Sigma_f)$.

(ii) As noted above since $n \geq 1$, it follows from [2, Theorems 1.5(i), 1.9] that $\mathcal{O}_f(f, \Sigma_f)$ is contractible. Then from the exact sequence of homotopy groups of the fibration k we obtain that for $i \geq 2$ every k_i is an isomorphism, and k_1 is a monomorphism. Lemma 8 is proved. \square

Remark 9. In general the covering map $\mathcal{F}(f) \rightarrow \mathcal{F}_n(\text{Int}M)$ is not *regular*, i.e., $\pi_1 \mathcal{O}_f(f) \approx \pi_1 \mathcal{F}(f)$ is not a normal subgroup of $\mathcal{B}_n(M) = \pi_1 \mathcal{F}_n(\text{Int}M)$.

Remark 10. Theorem 2 does not answer the question whether $\mathcal{O}_f(f)$ has the homotopy type of a *finite* CW-complex. Indeed, since M is compact, it follows from (3.1) that $\mathcal{B}_n(M)$ can be regarded as an open cellular (i.e. consisting of full cells) subset of a finite CW-complex $\coprod_n M/S_n$. Therefore if the covering map $\mathcal{F}(f) \rightarrow \mathcal{F}_n(\text{Int}M)$ is an infinite sheet covering, i.e., $\pi_1 \mathcal{O}_f(f)$ has an infinite index in $\mathcal{B}_n(M)$, then we obtain a priori an infinite cellular subdivision of $\mathcal{F}(f)$. On the other hand, as noted above, for a generic Morse map $f : M \rightarrow P$ a finiteness of the homotopy type of $\mathcal{O}_f(f)$ follows from [2].

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